Lec 7

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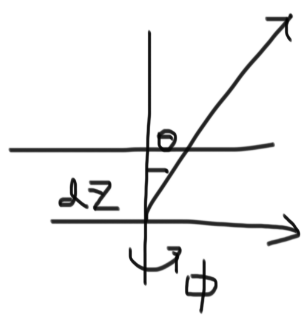
1 Eddington Approximation



Set up: Assume plane parallel atmosphere: no ϕ dependence

$$\int_{4\pi} = \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi = \int_{-1}^{+1} d\mu = \int_0^{2\pi} d\phi = 2\pi \int_{-1}^{1} d\mu$$

$$\text{where } \frac{d\cos\theta}{d\theta} = -\sin\theta \Rightarrow u = \cos\theta,\ du = \sin\theta d\theta$$

$$\Rightarrow J_\nu = \frac{\int I_\nu d\Omega}{\int d\Omega} = \frac{\int_{-1}^{1} d\mu \int_0^{2\pi} I_\nu d\phi}{4\pi} = \frac{2\pi \int_{-1}^{1} d\mu I_\nu}{4\pi} = \frac{1}{2}\int_{-1}^{1} d\mu I_\nu$$

$$H_\nu^i = \frac{\int_{-1}^{1} d\mu \int_0^{2\pi} d\phi I_\nu \hat{n}^i}{4\pi} = \frac{1}{2}\int_{-1}^{1} \mu I_\nu d\mu$$

Consider J_ν as the zeroth moment of radiation field and H_ν^I as the first moment where one direction is added. Now the second moment:

$$K_\nu^{ij} = \frac{\int_{-1}^{1} d\mu \int_0^{2\pi} d\phi I_\nu \hat{n}^i \hat{n}^j}{4\pi} = \frac{1}{2}\int_{-1}^{1} \mu^2 I_\nu d\mu$$

Examine all three moments:

$$J_\nu = \frac{1}{2}\int_{-1}^{1} d\mu I_\nu \quad (1)$$

$$H_\nu^i = \frac{1}{2}\int_{-1}^{1} d\mu I_\nu \quad (2)$$

$$K_\nu^{ij} = \frac{1}{2}\int_{-1}^{1} \mu^2 I_\nu d\mu \quad (3)$$

$$\mu \frac{dI_\nu}{d\tau} = -(I_\nu - S_\nu)$$

$$\underbrace{\frac{1}{2}\int_{-1}^{1} d\mu\mu \frac{dI_\nu}{\tau}}_{\frac{dH_\nu}{d\tau}} = \underbrace{-\frac{1}{2}\int_{-1}^{1} d\mu I_\nu}_{-J_\nu} + \underbrace{\frac{1}{2}\int_{-1}^{1} d\mu S_\nu}_{S_\nu}$$

$$\boxed{\begin{aligned} \Rightarrow \frac{dH_\nu}{d\tau} &= -J_\nu + S_\nu && (4) \\ \frac{dK_\nu}{d\tau} &= \frac{1}{3}\frac{dJ_\nu}{d\tau} = -H_\nu && (5) \end{aligned}}$$

Assume at depths $\gg$ effective mfp

$$I_\nu(\mu) \approx a + b\mu + \mathcal{O}(\mu^2)$$

where a, b are constraints and higher order order of μ are ignored

$$J_\nu = \frac{1}{2}\int_{-1}^{1} I_\nu d\mu = \frac{1}{2}\int_{-1}^{1} a d\mu - \cancelto{0}{\frac{1}{2}\int_{-1}^{1} b\mu d\mu} = a$$

$$H_\nu = \cancelto{0}{\frac{1}{2}\int_{-1}^{1} a\mu d\mu} + \frac{1}{2}\int_{-1}^{1} b\mu^2 d\mu = \frac{b}{3}$$

$$K_\nu = \frac{1}{2}\int_{-1}^{1} a\mu^2 d\mu + \cancelto{0}{\frac{1}{2}\int_{-1}^{1} b\mu^r d\mu} = \frac{a}{3}$$

$$\Rightarrow \boxed{K_\nu = \frac{J_\nu}{3}} \quad \text{(Eddington Approximation)}$$

Was the same as the result from isotropic radiation field

$$\frac{dK_\nu}{d\tau} = -H_\nu \Rightarrow \frac{1}{3}\frac{dJ_\nu}{d\tau} = -H_\nu$$

$$\overset{\frac{d}{d\tau}}{\Rightarrow} \frac{1}{3}\frac{d^2 J_\nu}{d\tau^2} = -\frac{dH_\nu}{d\tau} = -(-J_\nu + S_\nu) = J_\nu - S_\nu$$

$$\frac{1}{3}\frac{d^2 J_\nu}{d\tau^2} = J_\nu - (\epsilon_\nu S_{\nu,abs} + (1-\epsilon)J_\nu) = -\epsilon_\nu(S_{\nu,abs} - J_\nu)$$

Plug in Boundary conditions:

$$\left.\begin{array}{l} J_\nu(0) = J_{\nu,0},\ J_\nu(\infty) = S_{\nu,abs} \\ \text{assume } S_{\nu,abs} = \text{const} \end{array}\right\} \Rightarrow J_\nu = S_{\nu,abs} + (J_{\nu,0} - S_{\nu,abs})e^{-\tau\sqrt{3\epsilon_\nu}}$$

$$J_\nu \to S_{\nu,abs}$$

where $-\tau\sqrt{3\epsilon_\nu}$ is the effective optical depth modified by scattering.

2 Accelerating a charged particle - Poynting vector

Maxwell's eq

$$\text{Coulumb's law}\nabla \cdot \vec{D} = 4\pi\rho\text{Faraday's law}\nabla \times \vec{E} = \frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0$$

$$\text{Gen. Ampere's law}\nabla \times \vec{H} = -\frac{1}{c}\frac{\partial \vec{D}}{dt} + \frac{\partial 4\pi}{c}\vec{j}$$

where ρ: charge density, $\vec{j}$: current density

$$\vec{D} = \epsilon \vec{E}$$
$$\vec{B} = \mu \vec{H}$$

where ϵ is the dielectric constant, μ is the magnetic permeability, in vacuum

$$\epsilon = \mu = 1$$

$$\nabla \cdot \vec{E} = 4\pi\rho$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{e}}{\partial t} + \frac{4\pi}{c} + \frac{4\pi}{c}\vec{j}$$

Also lorentz force

$$\vec{F} = q(\vec{E} + \frac{1}{c}(\vec{v} \times \vec{B}))$$

Some vector algebra...we get charge conservation

$$\cancelto{0}{\nabla \cdot (\nabla \times \vec{B})} = \frac{1}{c}\nabla \cdot \frac{\partial \vec{e}}{\partial t} + \frac{4\pi}{c} + \frac{4\pi}{c}\nabla \cdot \vec{j}$$
$$0 = \frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot \vec{E} + \frac{4\pi}{c}\nabla \cdot \vec{j}$$
$$\Rightarrow -\frac{\partial \rho}{\partial t} = \nabla \cdot \vec{j}$$

In vacuum:

$$\nabla \cdot \vec{E} = 0$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{e}}{\partial t}$$

Can now substitute $\vec{B}$ and $\vec{E}$ freely. Also

$$\nabla \times (\nabla \times \vec{E}) = \nabla \times \left(\frac{1}{c}\frac{\partial \vec{B}}{\partial t}\right)$$

$$-\nabla^2 \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \vec{B})$$

$$\Rightarrow \nabla^2 \vec{E} = -\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t^2} = 0 \qquad \text{(Wave eq.)}$$

We define

$$\vec{E} = \hat{a}_1 E_0 e^{i\vec{k}\cdot\vec{r}-i\omega t}, \vec{B} = \hat{a}_2 B_0 e^{i\vec{k}\cdot\vec{r}-i\omega t}$$

where E_0, B_0 are complex constants, $\hat{a}_1, \hat{a}_2$ are unit vectors, $\vec{k} = k\vec{n}$ where k is the wave vector $2\pi/\lambda$ with unit $[cm^{-1}]$ and $\vec{n}$ is the unit vector in the direction of the wave propagation, $\vec{r}$ is the position, ω is the angular frequency $[s^{-1}]$.

Plug into maxwell, recall

$$\boxed{\nabla \cdot \psi\vec{A} = \psi(\nabla \cdot \vec{A}) + \vec{A} \cdot \nabla\psi}$$

$$\nabla \cdot \vec{E} = 0 \Rightarrow \nabla \cdot (\hat{a}_1 E_0 e^{i\vec{k}\cdot\vec{r}-i\omega t}) = 0$$

$$\Rightarrow \hat{a}_1 \vec{E}_0 \cdot \nabla(e^{i\vec{k}\cdot\vec{r}-i\omega t}) = 0$$

$$\hat{a}_1 E_0 \cdot i\vec{k} e^{i\vec{k}\cdot\vec{r}-i\omega t} = 0$$

$$\hat{a}_1 \cdot \vec{k} = 0 \tag{6}$$

$$\hat{a}_2 \cdot \vec{k} = 0 \tag{7}$$

$$\nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \Rightarrow \hat{a}_2 \vec{b}_0 \cdot \nabla(e^{i\vec{k}\cdot\vec{r}-i\omega t}) = \frac{1}{c}\frac{\partial}{\partial t}(\hat{a}_1 \vec{E}_0 \cdot \nabla(e^{i\vec{k}\cdot\vec{r}-i\omega t}))$$

$$i\vec{k} \times \hat{a}_2 B_0 = -\frac{i\omega}{c}\hat{a}_1 E_0 \tag{8}$$

$$i\vec{k} \times \hat{a}_1 E_0 = \frac{i\omega}{c}\hat{a}_2 B_0 \tag{9}$$

$$\Rightarrow E_0 = \frac{\omega}{kc}B_0,\ B_0 = \frac{\omega}{kc}E_0 \Rightarrow E_0 = \left(\frac{\omega}{kc}\right)^2 E_0 \Rightarrow \omega^2 = k^2c^2,\ B_0 = E_0$$

We can define (in vacuum, for nondispersive waves) phase velocity

$$\frac{\omega}{k} = c$$

group velocity

$$\frac{d\omega}{dk} = c$$

How much energy does EM wave carry?

$$\vec{F} = q\left(\frac{E}{+}\frac{\vec{v}}{c} \times \vec{B}\right)$$

$$\vec{f} = \rho\vec{E} + \frac{1}{c}\vec{j} \times \vec{B}, \text{ where}\rho = \lim_{\Delta V \to 0} \frac{\sum_i q_i}{\Delta V}, \vec{j} = \lim_{\Delta V \to 0} \frac{\sum_i q_i \vec{v}_i}{\Delta V}$$

Work?Power? done by E and B fields on a single particle:

$$\vec{v} \cdot \vec{F} = \vec{v} \cdot q\left(\frac{E}{+}\frac{\vec{v}}{c} \times \vec{B}\right) \to q\vec{v} \cdot E$$

Work/unit volume = rate of change of mechanical energy per unit volume:

$$\frac{dU_{mech}}{dt} = \vec{j} \cdot \vec{E}$$

Rewrite with various maxwell eq.

$$\frac{dU_{mech}}{dt} = \frac{1}{4\pi}\left(c(\nabla \times \vec{H}) \cdot \vec{E} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}\right)$$

$$= \frac{1}{4\pi}\left(c(\vec{H} \cdot (\nabla \times \vec{E}) - \nabla \cdot (\vec{E} \times \vec{H})) - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}\right)$$

$$= \frac{1}{4\pi}\left(c\left(\vec{H} - \frac{1}{c}\frac{\partial \vec{B}}{\partial t}\right) - c\nabla \cdot (\vec{E} \times \vec{H}) - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}\right)$$

$$\vec{j} \cdot \vec{E} + \frac{1}{8\pi}\frac{\partial}{\partial t}\left(\epsilon E^2 + \frac{B^2}{\mu}\right) = -\nabla \cdot \left(\frac{c}{4\pi}\vec{E} \times \vec{H}\right)$$

is the Poynting theorem. Now

$$U_{field} = \frac{1}{8\pi}\frac{\partial}{\partial t}\left(\epsilon E^2 + \frac{B^2}{\mu}\right) = U_E + U_B$$

$$\Rightarrow \vec{S} \equiv \left(\frac{c}{4\pi}\vec{E} \times \vec{H}\right)$$

$$\frac{dU_{mech}}{dt} = \underbrace{\frac{\partial}{\partial t}U_{field}}_{\text{rage of change of energy density}} - \underbrace{\nabla \cdot \vec{S}}_{\text{divergence of flux vector}}$$

In vacuum:

$$\frac{dU_{mech}}{dt} = -\frac{\partial}{\partial dt}\left(\frac{1}{8\pi}(E^2 + B^2)\right) - \frac{c}{4\pi}\nabla \cdot (\vec{E} \times \vec{B})$$