Physics 224: Lecture 2 notes
Spring 2018

General collisional rate definition:

Collisions control many of the key processes in the ISM (see slides for a non-exhaustive list). We can set up a very general framework, where the collision involves $A + B \rightarrow$ products. For different types of collisional processes this could be:

- elastic: scattering where for example $A + B \rightarrow A + B$, the particles exchange momentum in the collision.
- inelastic: scattering where for example $A^* + B \rightarrow A + B^*$, there are changes in internal energy of the particles as a result of the collision.
- chemical: where for example $A + B \rightarrow C$, the product is different than the initial colliders.

A simple way to think about the interactions is to take a particle A with a cross-section (imagine just a geometric cross section for simplicity) σ_A moving with velocity v_A. This would sweep up a volume of $\sigma_A v_A$ per second. Now imagine there are particles B with a density n_B that for the moment we treat as having very very small cross sections. The number of collisions of A with B per second would be $n_B \sigma_A v_A$. To make this more general we can say that the cross-section for collisions of A and B is σ_{AB} so we don't ignore the fact that B has some cross-section as well. Likewise, we can say the velocity of A relative to B is v_{AB}. If the density of A is n_A, we get the following:

$$\text{collisions per second per volume} = n_A n_B \sigma_{AB} v_{AB} \tag{1}$$

So far this has assumed that all particles have the same v_{AB}, which is not usually the case. Particles have a distribution of velocities given by a Maxwellian (even in the ISM there are enough collisions to generally maintain a Maxwellian distribution):

$$f_v dv = 4\pi \left(\frac{\mu}{2\pi kT}\right)^{3/2} e^{-\mu v^2/2kT} v^2 dv \tag{2}$$

where μ is the reduced mass $m_A m_B/(m_A + m_B)$.

To get the reaction rate per volume correct, we need to take into account the distribution of velocities. We can do this by defining the "two body collisional rate coefficient" $\langle \sigma v \rangle_{AB}$ so that the reaction rate per volume $= n_A n_B \langle \sigma v \rangle_{AB}$ and:

$$\langle \sigma v \rangle_{AB} = \int_0^\infty \sigma_{AB}(v) v f_v dv \tag{3}$$

This basically is weighting the cross-section (which can depend on velocity) by the distribution of velocities in the Maxwellian.

Types of collisions:

There are three basic types of collisions we will discuss:

- "hard sphere" collisions
- charged-neutral particle collisions
- charged-charged particle collisions

It makes sense to separate these this way because they are categorized in order of how important the long range forces between the particles are in the collision. For "hard sphere" collisions, the long range forces are unimportant. We can think of this like two dust grains running into each other–they have physical cross sections and generally don't interact until they run into each other. The charged-neutral interactions have some long-range interaction we can't ignore in determining their rate. For charged-charged interactions, the long range forces are very important.

"Hard sphere": Lets talk first about "hard sphere" collisions. Imagine you have two particles with different radii r_A and r_B. Their cross section will be $\sigma_{AB} = \pi(r_A+r_B)^2$. What you'll notice here is that the cross section is independent of energy or velocity. We can put this back in our expression above for the collisional rate coefficient:

$$\langle \sigma v \rangle_{AB} = \int_0^\infty \sigma_{AB} v 4\pi \left(\frac{\mu}{2\pi kT}\right)^{3/2} e^{-\mu v^2/2kT} v^2 dv \tag{4}$$

$$= 4\pi \left(\frac{\mu}{2\pi kT}\right)^{3/2} \sigma_{AB} \int_0^\infty v^3 e^{-\mu v^2/2kT} dv \tag{5}$$

To solve this, it is easiest to transform to energy units, using the fact that the probability of finding a particle in a given velocity range of $v \rightarrow v + dv$ is the same as in the equivalent energy range $E \rightarrow E + dE$ since E is a monotonic function of v. It also helps to switch variables so $x = E/kT$. Doing both of those things you can work out that:

$$\langle \sigma v \rangle_{AB} = \sigma_{AB} \sqrt{\frac{8kT}{\pi\mu}} \int_0^\infty x e^{-x} (xkT) dx \tag{6}$$

$$\langle \sigma v \rangle_{AB} = \sigma_{AB} \sqrt{\frac{8kT}{\pi\mu}} \tag{7}$$

since the integral goes to 1.

This tells us that the collisional rate coefficient depends on $T^{1/2}$. We can look at one important ISM example—the collisions between neutral particles. At large distances the forces between two neutral particles are not strong, but when they get close enough that their electron clouds begin to interact, it quickly becomes very strong. This makes the collisions between neutrals behave essentially like hard spheres running into each other. We can take the radius of the spheres to be ~ 1 Å so $\sigma_{AB} = \pi(2\text{Å})^2 \sim 1.2 \times 10^{-15}$ cm^2. Putting this into our hard sphere collisional rate coefficient equation, we can calculate:

$$\langle \sigma v \rangle_{AB} = 1.81 \times 10^{-10} \left(\frac{T}{100\text{K}} \right)^{1/2} \left(\frac{\mu}{m_H} \right)^{-1/2} \left(\frac{r_A + r_B}{2\text{Å}} \right)^2 \text{cm}^3\text{s}^{-1}. \tag{8}$$

"Charged-Neutral Collisions": In collisions of charged and neutral particles, the charged particle will induce a electric dipole moment in the neutral particle, which then creates a $1/r^4$ potential. Lets imagine an ion with velocity v and charge Ze interacting with a neutral particle at rest. In $1/r^4$ potentials, you can define a critical impact parameter b_0 for an encounter between the charged and neutral particles were at $r < b_0$ the charged particle experiences a very large deflection from its trajectory. At $r > b_0$ the deflection is much smaller. The critical impact parameter depends on the initial kinetic energy of the charged particle in the center of mass frame E_{cm}. With these parameters:

$$b_0 = \left(\frac{2\alpha_N Z^2 e^2}{E_{cm}} \right)^{1/4} \tag{9}$$

where α_N is the polarizability of the neutral particle, which is typically $\alpha_N \sim$ a few $\times a_0^3$ (where a_0 is the Bohr radius).

Using this critical impact parameter to define our cross section, $\sigma = \pi b_0^2$ we find:

$$\sigma = \pi b_0^2 = 2\pi Ze \left(\frac{\alpha_N}{\mu} \right)^{1/2} \frac{1}{v} \tag{10}$$

since $E_{cm} = \mu v^2/2$. Putting this into our expression:

$$\langle \sigma v \rangle_{AB} = \int_0^\infty \sigma_{AB}(v) v f_v dv \tag{11}$$

$$= \int_0^\infty 2\pi Ze \left(\frac{\alpha_N}{\mu} \right)^{1/2} \frac{1}{v} v f_v dv \tag{12}$$

$$= 2\pi Ze \left(\frac{\alpha_N}{\mu} \right)^{1/2} \int_0^\infty f_v dv \tag{13}$$

$$= 2\pi Ze \left(\frac{\alpha_N}{\mu} \right)^{1/2} \tag{14}$$

Since the integral of the probability distribution function over all velocities is 1 (note: this is where I messed up with the integral! - there is f_v in there still), we end up with a rate coefficient that doesn't depend on energy or temperature of the particles. This means that even in cold regions, when there are charged particles and neutrals, this rate will be important.

"Charged-Charged Collisions": In these type of collisions the long range forces are not negligible so we have to be specific in how we ask questions about cross sections. For instance, we can ask for what collision impact parameter does a charged particle moving close to another charged particle gain enough energy to eject an electron (i.e. for collisional ionization). To work out these types of interactions we will use the "impact approximation" - this problem can be solved directly but as the book says, the integrals are tedious.

Imagine a particle with charge Z_2 moving by a particle Z_1 with its closest approach at a distance b. See the diagram in your textbook for this set up. We can calculate the force between the two particles at every point along its path as:

$$F_\perp = \frac{Z_1 Z_2 e^2}{(b/cos\theta)^2} cos\theta \tag{15}$$

The amount of momentum gained perpendicular to the direction of motion is then the integral of this force:

$$\Delta p_\perp = \int_{-\infty}^{\infty} F_\perp dt \tag{16}$$

Switching from dt to $d\theta$ and doing the integral you'll find:

$$\Delta p_\perp = 2\frac{Z_1 Z_2 e^2}{b v_1} \tag{17}$$

From here you can ask specific questions, like when is the gain of kinetic energy from the interaction greater than the ionization potential of the collider. We can also use this set up to address an important collisional process in the ISM: how electrons or other charged particles distribute their energy among the other gas particles (i.e. after an electron is ejected by the photoelectric effect from a dust grain, how does it go on to heat the gas?). To look into this we can imagine the projectile with charge $Z_1 e$ moving through a field of charges $Z_2 e$. The projectile will get many individual momentum kicks from collisions and this results in a random walk in the average perpendicular momentum of the particle. We will start up here in the next lecture!